UNIVERSITY OF MASSACHUSETTS
Dept. of Electrical & Computer Engineering

Digital Computer Arithmetic
ECE 666

Part 7a
Fast Division - I

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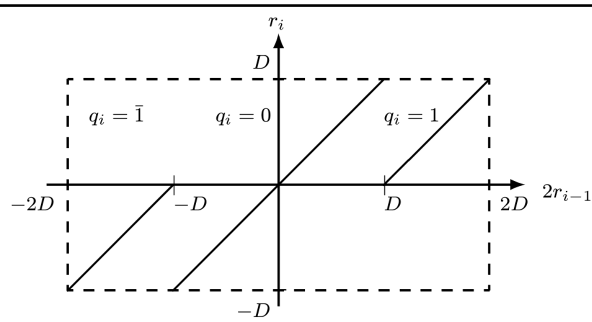
Fast Division - SRT Algorithm

- 2 approaches:
  - First - conventional - uses add/subtract+shift, number of operations linearly proportional to word size n
  - Second - uses multiplication, number of operations logarithmic in n, but each step more complex
  - SRT - first approach
- Most well known division algorithm - named after Sweeney, Robertson, and Tocher
- Speed up nonrestoring division (n add/subtracts) - allows 0 as a quotient digit - no add/subtract:

$$q_i = \begin{cases} 1 & \text{if } \; 2r_{i-1} \geq D \\ 0 & \text{if } \; -D \leq 2r_{i-1} < D \\ \bar{1} & \text{if } \; 2r_{i-1} < -D \end{cases}$$

$$r_i = 2r_{i-1} - q_i \cdot D$$

Modified Nonrestoring Division



- **Problem**: full comparison of $2r_{i-1}$ with either D or -D required
- **Solution**: restricting D to normalized fraction $1/2 \le |D| < 1$
- Region of $2r_{i-1}$ for which $q_i=0$ reduced to

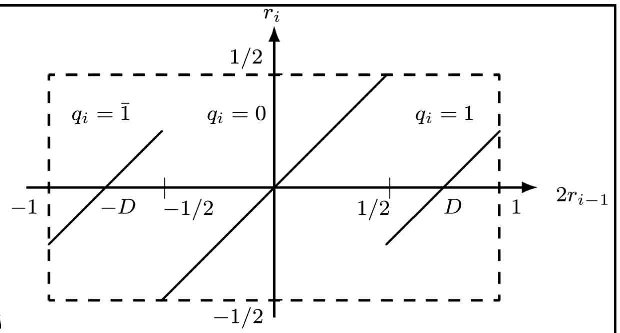
$$-D \le -\frac{1}{2} \le 2r_{i-1} < \frac{1}{2} \le D$$

Modified Nonrestoring → SRT

- **Advantage**: Comparing partial remainder $2r_{i-1}$ to 1/2 or -1/2, not D or -D
- Binary fraction in two's complement representation
  - * ≥ 1/2 if and only if it starts with 0.1
  - * ≤ -1/2 if and only if it starts with 1.0
- Only 2 bits of $2r_{i-1}$ examined - not full comparison between $2r_{i-1}$ and D
  - * In some cases (e.g., dividend X>1/2) - shifted partial remainder needs an integer bit in addition to sign bit - 3 bits of $2r_{i-1}$ examined
- Selecting quotient digit:

$$q_i = \begin{cases} 1 & \text{if } \; 2r_{i-1} \ge 1/2 \\ 0 & \text{if } \; -1/2 \le 2r_{i-1} < 1/2 \\ \bar{1} & \text{if } \; 2r_{i-1} < -1/2. \end{cases}$$

SRT Division Algorithm



- Quotient digits selected so $|r_i| \le |D| \Rightarrow$ final remainder $< |D|$
- Process starts with normalized divisor - normalizing partial remainder by shifting over leading 0's/1's if positive/negative
- Example:
  - * $2r_{i-1}$=0.001xxxx (x - 0/1); $2r_{i-1}$<1/2 - set q_i=0, $2r_i$=0.01xxxx and so on
  - * $2r_{i-1}$=1.110xxxx; $2r_{i-1}$>-1/2 - set q_i=0, $2r_i$=1.10xxxx
- SRT is nonrestoring division with normalized divisor and remainder

Extension to Negative Divisors

$$q_i = \begin{cases} 0 & \text{if } |2r_{i-1}| < 1/2 \\ 1 & \text{if } |2r_{i-1}| \ge 1/2 \ \& \ r_{i-1} \text{ and } D \text{ have the same sign} \\ \bar{1} & \text{if } |2r_{i-1}| \ge 1/2 \ \& \ r_{i-1} \text{ and } D \text{ have opposite signs} \end{cases}$$

- Example: Dividend X=(0.0101)₂ =5/16 Divisor D=(0.1100)₂ =3/4

$r_0 = X$		0	.0	1	0	1	
$2r_0$		0	.1	0	1	0	$\ge 1/2$ set $q_1 = 1$
Add $-D$	+	1	.0	1	0	0	
r_1		1	.1	1	1	0	
$2r_1 = r_2$		1	.1	1	0	0	$\ge -1/2$ set $q_2 = 0$
$2r_2 = r_3$		1	.1	0	0	0	$\ge -1/2$ set $q_3 = 0$
$2r_3$		1	.0	0	0	0	$< -1/2$ set $q_4 = \bar{1}$
Add D	+	0	.1	1	0	0	
r_4		1	.1	1	0	0	negative remainder & positive X
Add D	+	0	.1	1	0	0	correction
r_4		0	.1	0	0	0	corrected final remainder

- Before correction Q=0.100$\bar{1}$ - minimal SD repr. of Q=0.0111 - minimal number of add/subtracts
- After correction, Q = 0.0111-ulp = 0.0110₂ = 3/8 ; final remainder = $1/2 \cdot 2^{-4}$ =1/32

Example

- $X=(0.00111111)_2=63/256$ $D=(0.1001)_2=9/16$

$r_0 = X$		0	.0	0	1	1	1	1	1	1	
$2r_0$		0	.0	1	1	1	1	1	1	0	$< 1/2$ set $q_1 = 0$
$2r_1$		0	.1	1	1	1	1	1	0	0	$\geq 1/2$ set $q_2 = 1$
Add $-D$	+	1	.0	1	1	1					
r_2		0	.0	1	1	0	1	1	0	0	
$2r_2$		0	.1	1	0	1	1	0	0	0	$\geq 1/2$ set $q_3 = 1$
Add $-D$	+	1	.0	1	1	1					
r_3		0	.0	1	0	0	1	0	0	0	
$2r_3$		0	.1	0	0	1	0	0	0	0	$\geq 1/2$ set $q_4 = 1$
Add $-D$	+	1	.0	1	1	1					
r_4		0	.0	0	0	0	0	0	0	0	zero final remainder

- $Q = 0.0111_2 = 7/16$ - not a minimal representation in SD form
- Conclusion: Number of add/subtracts can be reduced further

Properties of SRT

- Based on simulations and analysis:
- 1.Average "shift"=2.67 - n/2.67 operations for dividend of length n
  - 24/2.67 ~ 9 operations on average for n=24
- 2.Actual number of operations depends on divisor D - smallest when 17/28 ≤ D ≤ 3/4 - average shift of 3
- If D out of range (3/5 ≤ D ≤ 3/4) - SRT can be modified to reduce number of add/subtracts
- 2 ways to modify SRT

Two Modifications of SRT

- **Scheme 1:** In some steps during division -
  - * If D too small - use a multiple of D like 2D
  - * If D too large - use D/2
  - * Subtracting 2D (D/2) instead of D - equivalent to performing subtraction one position earlier (later)
- **Motivation for Scheme 1:**
  - * Small D may generate a sequence of 1's in quotient one bit at a time, with subtract operation per each bit
  - * Subtracting 2D instead of D (equivalent to subtracting D in previous step) may generate negative partial remainder, generating sequence of 0's as quotient bits while normalizing partial remainder
- **Scheme 2:** Change comparison constant K=1/2 if D outside optimal range - allowed because ratio D/K matters - partial remainder compared to K not D

Example - Scheme 1 (Using 2D)

- Same as previous example -
- $X=(0.00111111)_2=63/256$ $D=(0.1001)_2=9/16$

$r_0 = X$			0	.0	0	1	1	1	1	1	1	
$2r_0$			0	.0	1	1	1	1	1	1	0	$< 1/2$ set $q_1 = 0$
$2r_1$		0	0	.1	1	1	1	1	1	0	0	subtract $2D$
Add $-2D$	+	1	0	.1	1	1						instead of D
r_2		1	1	.1	1	0	1	1	1	0	0	set $q_1 = 1$ and $q_2 = 0$
$2r_2$			1	.1	0	1	1	1	0	0	0	set $q_3 = 0$
$2r_3$			1	.0	1	1	1	0	0	0	0	$\le -1/2$ set $q_4 = \bar{1}$
Add D	+		0	.1	0	0	1					
r_4			0	.0	0	0	0	0	0	0	0	zero final remainder

- $Q = 0.100\bar{1}_2 = 7/16$ - minimal SD representation

Scheme 1 (Using D/2)

- Large D - one 0 in sequence of 1's in quotient may result in 2 consecutive add/subtracts instead of one
- Adding D/2 instead of D for last 1 before the single 0 - equivalent to performing addition one position later - may generate negative partial remainder
- Allows properly handling single 0
- Then continue normalizing partial remainder until end of sequence of 1's

Example

- $X=(0.01100)_2=3/8$; $D=(0.11101)_2=29/32$
- Correct 5-bit quotient - $Q=(0.01101)_2=13/32$
- Applying basic SRT algorithm - $Q=0.10\bar{1}1\bar{1}$ - single 0 not handled efficiently
- Using multiple D/2 -

$r_0 = X$		0	.0	1	1	0	0		
$2r_0$		0	.1	1	0	0	0		$\geq 1/2$ set $q_1 = 1$
Add $-D$	+	1	.0	0	0	1	1		
r_1		1	.1	1	0	1	1		
$2r_1$		1	.1	0	1	1	0		set $q_2 = 0$
$2r_2$		1	.0	1	1	0	0	0	add $D/2$ $q_3 = \bar{1}$)
Add $D/2$	+	0	.0	1	1	1	0	1	instead of D
r_3		1	.1	1	0	1	0	1	set $q_3 = 0$ and
$2r_3$		1	.1	0	1	0	1		$q_4 = \bar{1}$
$2r_4$		1	.0	1	0	1	0		$\leq -1/2$ set $q_5 = \bar{1}$
Add D	+	0	.1	1	1	0	1		
r_5		0	.0	0	1	1	1		final remainder $= 7/32 \cdot 2^{-5}$

- $Q=(0.100\bar{1}\bar{1})_2=13/32$ - single 0 handled properly

Implementing Scheme 1

- Two adders needed
  - One to add or subtract D
  - Second to add/subtract 2D if D too small (starts with 0.10 in its true form) or add/subtract D/2 if D too large (starts with 0.11)
- Output of primary adder used, unless output of alternate adder has larger normalizing shift
- Additional multiples of D possible - 3D/2 or 3D/4
- Provide higher overall average shift - about 3.7 - but more complex implementation

Modifying SRT - Scheme 2

- For $K=1/2$, ratio D/K in optimal range $3/5 \leq D \leq 3/4$ is
  $6/5 \leq D/K = D/(1/2) \leq 3/2$ or
  $(6/5)K \leq D \leq (3/2)K$
- If D not in optimal range for $K=1/2$ - choose a different comparison constant K
- Region $1/2 \leq |D| < 1$ can be divided into 5 (not equally sized) sub-regions
- Each has a different comparison constant K_i

Division into Sub-regions

1/2	9/16	5/8	3/4	15/16	1
.1000	.1001	.1010	.1100	.1111	1.0

$K_1 = 3/8$	$K_2 = 7/16$	$K_3 = 1/2$	$K_4 = 5/8$	$K_5 = 3/4$
.0110	.0111	.1000	.1010	.1100

- 4 bits of divisor examined for selecting comparison constant
- It has only 4 bits compared to 4 most significant bits of remainder
- Determination of sub-regions for divisor and comparison constants - numerical search
- Reason: Both are binary fractions with small number of bits to simplify division algorithm

Example

- $X=(0.00111111)_2=63/256$; $D=(0.1001)_2=9/16$
- Appropriate comparison constant - $K_2=7/16=0.0111_2$
- If remainder negative - compare to two's complement of $K_2 = 1.1001_2$

$r_0 = X$		0	.0	0	1	1	1	1	1	1	
$2r_0$		0	.0	1	1	1	1	1	1	0	≥ 0.0111 set $q_1 = 1$
Add $-D$	+	1	.0	1	1	1					
r_1		1	.1	1	1	0	1	1	1	0	
$2r_1 = r_2$		1	.1	1	0	1	1	1	0	0	≥ 1.1001 set $q_2 = 0$
$2r_2 = r_3$		1	.1	0	1	1	1	0	0	0	≥ 1.1001 set $q_3 = 0$
$2r_3$		1	.0	1	1	1	0	0	0	0	< 1.1001 set $q_4 = \bar{1}$
Add D	+	0	.1	0	0	1					
r_4		0	.0	0	0	0	0	0	0	0	zero final remainder

- $Q=0.100\bar{1}=0.0111_2=7/16$ - minimal SD form