UNIVERSITY OF MASSACHUSETTS
Dept. of Electrical & Computer Engineering

Digital Computer Arithmetic
ECE 666

Part 4-A
Floating-Point Arithmetic

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Preliminaries - Representation

- Floating-point numbers - provide a dynamic range of representable real numbers without having to scale the operands
- Representation - similar to scientific notation
- Two parts - significand (or mantissa) M and exponent (or characteristic) E
- The floating-point number F represented by the pair (M,E) has the value -

$$F = M\beta^E \quad (\beta \text{ - base of exponent})$$

- Base - common to all numbers in a given system - implied - not included in the representation of a floating point number

Preliminaries - Precision

- n bits partitioned into two parts - significand M and exponent E
- n bits - 2^n different values
- Range between smallest and largest representable values increases ⇒ distance between any two consecutive values increases
  - Floating-point numbers sparser than fixed-point numbers - lower precision
- Real number between two consecutive floating-point numbers is mapped onto one of the two
  - A larger distance between two consecutive numbers results in a lower precision of representation

Formats

- Significand M and exponent E - signed quantities
- Exponent - usually a signed integer
- Significand - usually one of two:
  - pure fraction, or * a number in the range [1, 2) (for β=2)
- Representing negative values - can be different
- Until 1980 - no standard - every computer system had its own representation method
  - transporting programs/data between two different computers was very difficult
- IEEE standard 754 is now used in most floating-point arithmetic units - details later
- Few computer systems use formats differing in partitioning of the n bits, representation of each part, or value of the base β

Significand Field

- Common case - signed-magnitude fraction
- Floating-point format - sign bit S, e bits of exponent E, m bits of unsigned fraction M (m+e+1=n)

S	Exponent E	Unsigned Significand M

- Value of (S,E,M) : $F = (-1)^S \cdot M \cdot \beta^E$
  ($(-1)^0=1$; $(-1)^1=-1$)
- Maximal value - Mmax = 1-ulp
- ulp -Unit in the last position - weight of the least-significant bit of the fractional significand
- Usually - not always - $ulp=2^{-m}$

The Base β

- β is restricted to 2^k (k=1,2,…) - simplifies decreasing significand and increasing exponent (and vice versa) at the same time
- Necessary when an arithmetic operation results in a significand larger than Mmax = 1-ulp - significand is reduced and exponent increased - value remains unchanged
- Smallest increase in E is 1

$$M \cdot \beta^E = (M/\beta) \cdot \beta^{E+1}$$

- M/β - a simple arithmetic shift right operation if β is an integral power of radix
- If β=r=2 - shifting significand to the right by a single position must be compensated by adding 1 to exponent

Example

- Result of an arithmetic operation - $01.10100 \cdot 2^{100}$ - significand larger than Mmax
- Significand reduced by shifting it one position to the right, exponent increased by 1
- New result - $0.11010 \cdot 2^{101}$
- If $\beta=2^k$ - changing exponent by 1 is equivalent to shifting significand by k positions
- Consequently - only k-position shifts are allowed
- If $\beta=4=2^2$
  $01.10100 \cdot 4^{010} = 0.01101 \cdot 4^{011}$

Normalized Form

- Floating point representation not unique - $0.11010 \cdot 2^{101} = 0.01101 \cdot 2^{110}$
- With E=111 - significand=0.00110 - loss of a significant digit
- Preferred representation - one with no leading zeros - maximum number of significant digits - normalized form
- Simplifies comparing floating-point numbers - a larger exponent indicates a larger number; significands compared only for equal exponents
- For $\beta=2^k$ - significand normalized if there is a nonzero bit in the first k positions
- Example: Normalized form of $0.00000110 \cdot 16^{101}$ is $0.01100000 \cdot 16^{100}$

Range of Normalized Fractions

- Range of significand is smaller than [0,1-ulp]
- Smallest and largest allowable values are
- Mmin = 1/β ; Mmax = 1-ulp
- Range of normalized fractions does not include the value zero - a special representation is needed
- A possible representation for zero - M=0 and any exponent E
- E=0 is preferred - representation of zero in floating-point is identical to representation in fixed-point
  - Execution of a test for zero instruction simplified

Representation of Exponents

- Most common representation - biased exponent
- $E = E^{true} + bias$ (bias - constant; E^{true} - the true value of the exponent represented in two's complement)
- Exponent field - e bits ; range: $-2^{e-1} \leq E^{true} \leq 2^{e-1} - 1$
- Bias usually selected as magnitude of most negative exponent 2^{e-1} $0 \leq E \leq 2^e - 1$
- Exponent represented in the excess 2^{e-1} method
- Advantages:
  - When comparing two exponents (for add/subtract operations) - sign bits ignored; comparison like unsigned numbers
  - Floating-points with S,E,M format are compared like binary integers in signed-magnitude representation
  - Smallest representable number has the exponent 0

Example: Excess 64

- e=7
- Range of exponents in two's complement representation is $-64 \le E^{true} \le 63$
- 1000000 and 0111111 represent -64 and 63
- When adding bias 64, the true values -64 and 63 are represented by 0000000 and 1111111
- This is called: excess 64 representation
- Excess 2^{e-1} representation can be obtained by
  - Inverting sign bit of two's complement representation, or
  - Letting the values 0 and 1 of the sign bit indicate negative and positive numbers, respectively

Range of Normalized Floating-Point Numbers

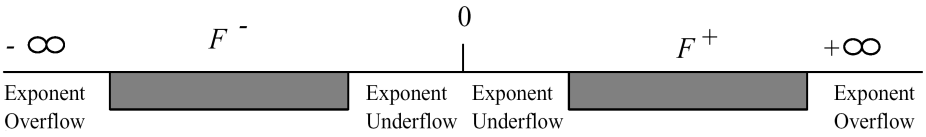
- Identical subranges for positive (F^+) and negative (F^-) numbers:

$$M_{min} \cdot \beta^{E_{min}} \le F^+ \le M_{max} \cdot \beta^{E_{max}}$$

  - (Emin, Emax - smallest,largest exponent)
- An exponent larger than Emax / smaller than Emin must result in an exponent overflow/underflow indication
- Significand normalized - overflow reflected in exponent
- Ways of indicating overflow:
  - Using a special representation of infinity as result
  - stopping computation and interrupting processor
  - setting result to largest representable number
- Indicating underflow:
  - Representation of zero is used for result and an underflow flag is raised - computation can proceed, if appropriate, without interruption

Range of Floating Point Numbers

- Zero is not included in the range of either F^+ or F^-



Example - IBM 370

- Short floating-point format - 32 bits ; β=16

S - sign bit	E - 7 bits, excess 64 exponent	M - 24 bits, unsigned fractional significand

$$F = (-1)^S \cdot M \cdot 16^{E-64}$$

- Emin, Emax represented by 0000000, 1111111 - value of -64, +63
- Significand - six hexadecimal digits
- Normalized significand satisfies -

$$M_{min} = 16^{-1} \leq M \leq M_{max} = 1 - 16^{-6} = 1 - 2^{-24}.$$

- Consequently,

$$F^+_{max} = (1 - 16^{-6}) \cdot 16^{63} \approx 7.23 \cdot 10^{75}$$

$$F^+_{min} = (16^{-1}) \cdot 16^{-64} \approx 5.4 \cdot 10^{-79}$$

Numerical Example - IBM 370

- **(S,E,M)=(C1200000)$_{16}$ in the short IBM format - first byte is (11000001)$_2$**
  - **Sign bit is S=1 - number is negative**
  - **Exponent is 41$_{16}$ and, bias is 64$_{10}$=40$_{16}$, E ^{true} =(41-40)$_{16}$=1**
  - **M=0.2$_{16}$, hence** $F = (-0.0010)_2 \cdot 16^1 = (-2)_{10}$.
- **Resolution of representation - distance between two consecutive significands -**

$$ulp = 16^{-6} = 2^{-24} \approx 0.6 \cdot 10^{-7}$$

  - **Short format has approximately 7 significant decimal digits**
- **For higher precision use the long floating-point format**

sign bit	7 bits - excess 64 exponent	56 bits - unsigned fractional significand

- **Range roughly the same, but resolution:**

$$ulp = 16^{-14} = 2^{-56} \approx 10^{-17}$$

  - **17 instead of 7 significant decimal digits**

Floating-Point Formats of Three Machines

	IBM/370	DEC/VAX	Cyber 70
Word length (double)	32 (64) bits	32 (64) bits	60 bits
Significand+{hidden bit}	24 (56) bits	$23+1$ $(55+1)$ bits	48 bits
Exponent	7 bits	8 bits	11 bits
Bias	64	128	1024
Base	16	2	2
Range of M	$\frac{1}{16} \le M < 1$	$\frac{1}{2} \le M < 1$	$1 \le M < 2$
Representation of M	Signed-magnitude	Signed-magnitude	One's complement
Approximate range	$16^{63} \approx 7 \cdot 10^{75}$	$2^{127} \approx 1.9 \cdot 10^{38}$	$2^{1023} \approx 10^{307}$
Approximate resolution	$2^{-24} \approx 10^{-7}$ (10^{-17})	$2^{-24} \approx 10^{-7}$ (10^{-17})	$2^{-48} \approx 10^{-14}$

Hidden Bit

- A scheme to increase the number of bits in significand to increase precision
- For a base of 2 the normalized significand will always have a leading 1 - can be eliminated, allowing inclusion of an extra bit
- The resolution becomes ulp=2^{-24} instead of 2^{-23}
- The value of a floating-point number (S,f,E) in short DEC format is

$$(-1)^S 0.1f \cdot 2^{E-128}$$

- f - the pattern of 23 bits in significand field

Hidden Bit - representation of zero

- A zero significand field (f=0) represents the fraction $0.10_2=1/2$
- If f=0 and E=0 - with a hidden bit, this may represent the value $0.1\ 2^{0-128} = 2^{-129}$
- The floating-point number f=E=0 also represents 0 - a representation without a hidden bit
- To avoid double meaning - E=0 reserved for representing zero - smallest exponent for nonzero numbers is E=1
- Smallest positive number in the DEC/VAX system - $F^+_{min} = \frac{1}{2}2^{1-128} = 2^{-128}$
- Largest positive number - $F^+_{max} = (1-2^{-24}) \cdot 2^{255-128} = (1-2^{-24}) \cdot 2^{127}$

Floating-Point Operations

- Execution depends on format used for operands
- Assumption: Significands are normalized fractions in signed-magnitude representation ; exponents are biased
- Given two numbers

$F_1 = (-1)^{S_1} \cdot M_1 \cdot \beta^{E_1 - bias}$; $F_2 = (-1)^{S_2} \cdot M_2 \cdot \beta^{E_2 - bias}$

- Calculate result of a basic arithmetic operation yielding

$$F_3 = (-1)^{S_3} \cdot M_3 \cdot \beta^{E_3 - bias}.$$

- Multiplication and division are simpler to follow than addition and subtraction

Floating-Point Multiplication

- Significands of two operands multiplied like fixed-point numbers - exponents are added - can be done in parallel
- Sign S_3 positive if signs S_1 and S_2 are equal - negative if not
- When adding two exponents $E_1 = E_1^{True} + bias$ and $E_2 = E_2^{true} + bias$: bias must be subtracted once
- For $bias = 2^{e-1}$ (100...0 in binary) - subtracting bias is equivalent to adding bias - accomplished by complementing sign bit
- If resulting exponent E_3 is larger than E_{max} / smaller than E_{min} - overflow/underflow indication must be generated

Multiplication - postnormalization

- Multiplying significands M_1 and M_2 - M_3 must be normalized
- $1/\beta \le M_1, M_2 < 1$ - product satisfies $1/\beta^2 \le M_1 \cdot M_2 < 1$
- Significand M_3 may need to be shifted one position to the left
- Achieved by performing one base- β left shift operation - k base-2 shifts for $\beta = 2^k$ - and reducing the exponent by 1
- This is called the postnormalization step
- After this step - exponent may be smaller than E_{min} - exponent underflow indication must be generated

Floating-Point Division

- Significands divided - exponents subtracted - bias added to difference $E_1 - E_2$
- If resulting exponent out of range - overflow or underflow indication must be generated
- Resultant significand satisfies $1/\beta \le M_1/M_2 < \beta$
- A single base-β shift right of significand + increase of 1 in exponent may be needed in postnormalization step - may lead to an overflow
- If divisor=0 - indication of division by zero generated - quotient set to $\pm\infty$
- If both divisor and dividend=0 - result undefined - in the IEEE 754 standard represented by NaN - not a number - also representing uninitialized variables and the result of $0 \cdot \infty$

Remainder in Floating-Point Division

- Fixed-point remainder - $R=X-QD$ (X, Q, D - dividend, quotient, divisor) - $|R| \leq |D|$ - generated by division algorithm (restoring or nonrestoring)
- Flp division - algorithm generates quotient but not remainder - $F1\ REM\ F2 = F1-F2 \cdot Int(F1/F2)$ ($Int(F1/F2)$ - quotient $F1/F2$ converted to integer)
- Conversion to integer - either truncation (removing fractional part) or rounding-to-nearest
- The IEEE standard uses the round-to-nearest-even mode - $|F1\ REM\ F2| \leq |F2|/2$
- $Int(F1/F2)$ as large as $\beta^{Emax-Emin}$ - high complexity
- Floating-point remainder calculated separately - only when required - for example, in argument reduction for periodic functions like sine and cosine

Floating-Point Remainder - Cont.

- Brute-force - continue direct division algorithm for $E1-E2$ steps
- Problem - $E1-E2$ can be much greater than number of steps needed to generate m bits of quotient's significand - may take an arbitrary number of clock cycles
- Solution - calculate remainder in software
- Alternative - Define a REM-step operation - $X\ REM\ F2$ - performs a limited number of divide steps (e.g., limited to number of divide steps required in a regular divide operation)
- Initial $X=F1$, then X=remainder of previous REM-step operation
- REM-step repeated until remainder $\leq F2/2$

Addition and Subtraction

- Exponents of both operands must be equal before adding or subtracting significands
- When E1=E2 - β^{E1} can be factored out and significands M1 and M2 can be added
- Significands aligned by shifting the significand of the smaller operand |E1-E2| base-β positions to the right, increasing its exponent, until exponents are equal
- E1≥E2 - $F_1 \pm F_2 = \left((-1)^{S_1} \cdot M_1 \pm (-1)^{S_2} \cdot M_2 \cdot \beta^{-(E_1 - E_2)}\right) \cdot \beta^{E_1 - bias}$
- Exponent of larger number not decreased - this will result in a significand larger than 1 - a larger significand adder required

Addition/Subtraction - postnormalization

- Addition - resultant significand M (sum of two aligned significands) is in range $1/\beta \le M < 2$
- If M>1 - a postnormalization step - shifting significand to the right to yield M3 and increasing exponent by one - is required (an exponent overflow may occur)
- Subtraction - Resultant significand M is in range $0 \le |M| < 1$ - postnormalization step - shifting significand to left and decreasing exponent - is required if $M < 1/\beta$ (an exponent underflow may occur)
- In extreme cases, the postnormalization step may require a shift left operation over all bits in significand, yielding a zero result

Example

- $F_1=(0.100000)_{16} \cdot 16^3$; $F_2=(0.FFFFFF)_{16} \cdot 16^2$
- Short IBM format ; calculate F_1-F_2

F_1	0.	1	0	0	0	0	0	·	16^3
F_2 aligned	0.	0	F	F	F	F	F	·	16^3
$F_1 - F_2$	0.	0	0	0	0	0	1	·	16^3
Postnormalization	0.	1	0	0	0	0	0	·	16^{-2}

- Significand of smaller number (F_2) is shifted to the right - least-significant digit lost
- Shift is time consuming - result is wrong

Example - Cont.

- Correct result (with "unlimited" number of significand digits)

F_1	0.	1	0	0	0	0	0	·	16^3
F_2 aligned	0.	0	F	F	F	F	F	·	16^3
$F_1 - F_2$	0.	0	0	0	0	0	1	·	16^3
Postnormalization	0.	1	0	0	0	0	0	·	16^{-3}

- Error (also called loss of significance) is

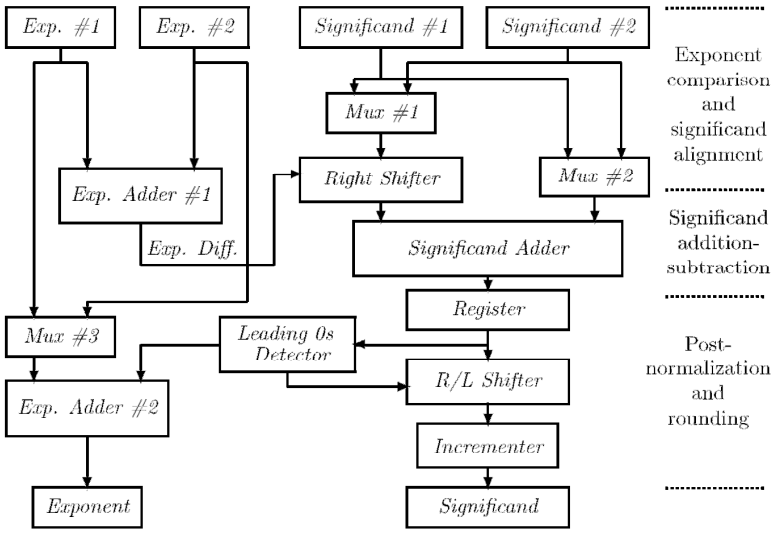
  $0.1 \cdot 16^{-2} - 0.1 \cdot 16^{-3} = 0.F \cdot 16^{-3}$

- Solution to problem - guard digits - additional digits to the right of the significand to hold shifted-out digits
- In example - a single (hexadecimal) guard digit is sufficient

Steps in Addition/Subtraction of Floating-Point Numbers

- **Step 1:** Calculate difference d of the two exponents - d=|E1 - E2|
- **Step 2:** Shift significand of smaller number by d base-β positions to the right
- **Step 3:** Add aligned significands and set exponent of result to exponent of larger operand
- **Step 4:** Normalize resultant significand and adjust exponent if necessary
- **Step 5:** Round resultant significand and adjust exponent if necessary

Circuitry for Addition/Subtraction



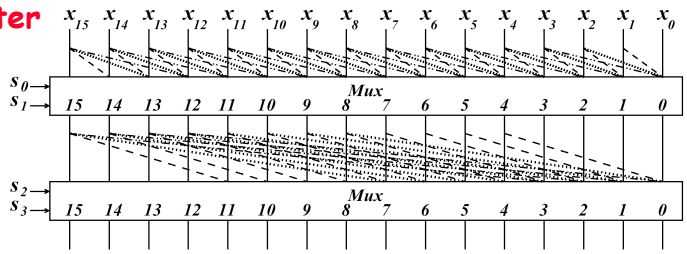
Shifters

- 1st shifter - right (alignment) shifts only ; 2nd shifter - right or left (postnormalization) shifts ; both perform large shifts (# of significand digits)
- Combinatorial shifter - generate all possible shifted patterns - only one at output according to control bits
  - Such shifters capable of circular shifts (rotates) - known as barrel shifters
  - Shift registers require a large and variable number of clock cycles, thus combinatorial shifters commonly used
- If implemented as a single level array - each input bit is directly connected to m (or more) output lines - conceptually simple design
- For m=53 (number of significand bits in IEEE double-precision format) - large number of connections (and large electrical load) - bad solution

Two levels Barrel Shifters

- first level shifts bits by 0, 1, 2 or 3 bit positions
- second level shift bits by multiples of 4 (0,4,8,...,52)
- shifts between 0 and 53 can be performed

- Radix-4 shifter
- 16 bits



- 1st level - each bit has 4 destinations ; 2nd level - each bit has 14 destinations - unbalanced

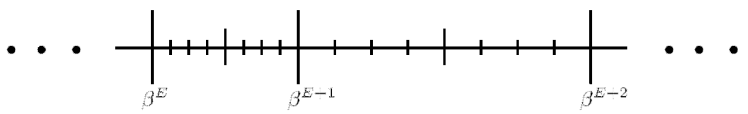
- Radix-8 shifter - 1st level shifts 0 to 7 bit positions ; 2nd level shifts by multiples of 8 (0,8,16,24,...,48)
  - 1st level - each bit has 8 destinations ; 2nd level - each bit has 7 destinations

Choice of Floating-Point Representation

- IEEE standard 754 commonly used - important to understand implications of a particular format
- Given n - total number of bits - determine
  - m - length of significand field
  - e - length of exponent field (m+e+1=n)
  - β - value of exponent base
- Representation error - error made when using a finite-length floating-point format to represent a high-precision real number
- x - a real number ; Fl(x) - its machine representation
- Goal when selecting format - small representation error
- Error can be measured in several ways

Measuring Representation Error

- Every real number x has two consecutive representations F1 and F2 satisfying $F1 \le x \le F2$
- Fl(x) can be set to either F1 or F2
- Fl(x)-x - absolute representation error
- $\delta(x)=(Fl(x)-x)/x$ - relative representation error
- If $F1=M\beta^E$ then $F2=(M+ulp)\beta^E$
- Maximum absolute error = half distance between F1 and F2 = $ulp\cdot\beta^E$ - increases as exponent increases



Measure of Representation Accuracy

- MRRE - maximum relative representation error - upper bound of δ(x)

$$\delta(x) \leq \frac{\frac{1}{2}\, ulp\ \beta^E}{M\beta^E} = \frac{1}{2}\frac{ulp}{M} \leq \frac{1}{2}\frac{ulp}{\frac{1}{\beta}} = \frac{1}{2}\, ulp \cdot \beta$$

- MRRE increases with exponent base β - decreases with ulp (or number of significand bits m)
- Good measure if operands uniformly distributed
- In practice - larger significands less likely to occur
- First digit of a decimal floating-point operand will most likely be a 1; 2 is the second most likely
- Operands follow the density function

$$\frac{1}{M \ln \beta}\ ; \qquad 1/\beta \leq M \leq 1$$

Different Accuracy Measure

- ARRE - average relative representation error
- Absolute error varies between 0 and 1/2 ulp·β^E
- Average absolute error is 1/4 ulp·β^E
- Relative error is 1/4 ulp/M

$$\text{ARRE} = \int_{\frac{1}{\beta}}^{1} \frac{1}{M \ln \beta}\frac{ulp}{4M}\, dM = \frac{\beta - 1}{\ln \beta}\frac{ulp}{4}$$

Range of Representation

- The range of the positive floating-point numbers - β^{Emax} - must be considered when selecting a floating-point format
- For a large range - increase β and/or number of exponent bits e
- Increasing β increases representation error
- Increasing e decreases m and increases ulp - higher representation error
- Trade-off between range and representation error

Range - Accuracy Trade-off

β	e	m	Range	MRRE	ARRE
2	9	22	$2^{2^8-1} = 2^{255}$	$0.5 \cdot 2^{-22} \cdot 2 = 2^{-22}$	$0.180 \cdot 2^{-21}$
4	8	23	$4^{2^7-1} = 2^{2^8-2} = 2^{254}$	$0.5 \cdot 2^{-23} \cdot 4 = 2^{-22}$	$0.135 \cdot 2^{-21}$
16	7	24	$16^{2^6-1} = 2^{2^8-4} = 2^{252}$	$0.5 \cdot 2^{-24} \cdot 16 = 2^{-21}$	$0.169 \cdot 2^{-21}$

- If several floating-point representations have same range - select smallest MRRE or ARRE
- If several representations have same MRRE (or ARRE) - select the largest range
- Example: 32-bit word - m+e=31 - all three representations have about the same range
- Using MRRE as measure - β=16 inferior to other two
- Using ARRE as measure - β=4 is best
- β=2 + hidden bit reduces MRRE and ARRE by a factor of 2 - the smallest representation error

Execution Time of Floating-Point Operations

- One more consideration when selecting a format
- Two time-consuming steps - aligning of significands before add/subtract operations ; postnormalization in any floating-point operation
- Observation - larger β - higher probability of equal exponents in add/subtract operations - no alignment ; lower probability that a postnormalization step needed
- No postnormalization in 59.4% of cases for β=2; 82.4% for β=16

Alignment shift	$\beta = 16$	$\beta = 2$
0	47.3%	32.6%
1	26.0%	12.1%
≥ 2	26.7%	55.3%

- This is of limited practical significance when a barrel shifter is used