UNIVERSITY OF MASSACHUSETTS
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Digital Computer Arithmetic
ECE 666

Part 2
Unconventional Number Systems

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Unconventional Fixed-Radix Number Systems

- Number system commonly used in arithmetic units - binary system with two's complement representation of negative numbers
- Other number systems have proven to be useful for certain applications -
- Negative radix number system
- Signed-digit number system
- Sign-logarithm number system
- Residue number system

Negative-Radix Number Systems

- The radix **r** of a fixed-radix system - usually a positive integer
- **r** can be negative - **r=-b** (b - a positive integer)
- Digit set - X_i = 0,1,...,b-1
- Value of n-tuple (X_{n-1},X_{n-2},...,X_0) -

$$X = \sum_{i=0}^{n-1} x_i \, (-\beta)^i.$$

- The weight W_i is -

$$w_i = \begin{cases} \beta^i & \text{if } i \text{ is even} \\ -\beta^i & \text{if } i \text{ is odd.} \end{cases}$$

Example - Negative-Decimal System

- Negative-radix number system with b=10 - nega-decimal system
- Three-digit nega-decimal numbers -
  - 192_{-10} =100-90+2=12 ; 012_{-10}=-10+2=-8
  - Largest positive value - 909_{-10}=909_{10}
  - Smallest value - 090_{-10}=-90_{10}
  - Asymmetric range - -90 £ X £ 909
  - Approximately 10 times more positives than negatives
- This is always true for odd values of n - opposite for even n
- Example - the range for n=4 is -9090 £ X £ 909

Negative-Radix Number Systems - Properties

- No need for a separate sign digit
- No need for a special method to represent negative numbers
- Sign of number is determined by the first nonzero digit
- No distinction between positive and negative number representations - arithmetic operations are indifferent to the sign of the numbers
- Algorithms for the basic arithmetic operations in the negative-radix number system are slightly more complex then their counterparts for the conventional number systems

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Example - Negative-Binary System

- Negative-radix numbers of length $n=4$, $b=2$ - nega-binary system
- Range - $-10_{10}=1010_{-2} \le X \le 0101_{-2}=+5_{10}$
- When adding nega-binary numbers - carry bits can be either positive or negative
- **Example:**

-8	+4	-2	+1	
0	0	1	0	-2
0	0	1	1	-1
1	1	0	1	-3

- Nega-binary - proposed for signal processing applications
- Algorithms exist for all arithmetic operations
- Did not gain popularity: Main reason - not better than the two's complement system

Signed-Digit Number Systems

- So far - digit set $\{0,...,r-1\}$
- In the signed-digit (SD) number system, digit set is $\{\overline{r-1},\overline{r-2},\dots,\bar{1},0,1,...,r-1\}$ ($\bar{i}$ = -i)
- No separate sign digit
- Example:
  - r=10, n=2 ; digits - $\{\bar{9},\bar{8},...,\bar{1},0,1,...,8,9\}$
  - Range - $\bar{9}\bar{9}$ £ X £ 99 - 199 numbers
  - 2 digits, 19 possibilities each - 361 representations - redundancy
  - $01=1\bar{9}=1$; $0\bar{2}=\bar{1}8=-2$
  - Representation of 0 (or 10) is unique
  - Out of 361 representations, 361-199=162 are redundant - 81% redundancy
  - Each number in range has at most two representations

Reducing Redundancy in Signed-Digit Number Systems

- Redundancy can be beneficial - but more bits needed
- Reducing redundancy - digit set restricted to

$$x_i \in \{\bar{a},\overline{a-1},\ldots,\bar{1},0,1,\ldots,a\} \quad \text{with} \quad \left\lceil \frac{r-1}{2} \right\rceil \le a \le r-1$$

  - é x ù - smallest integer larger than or equal to x

- To represent a number in a radix r system - at least r different digits are needed
- $\bar{a}$ £ X_i £ a - 2a+1 digits
- 2a+1 ³ r and $\left\lceil \frac{r-1}{2} \right\rceil \le a$

Example - SD Number System

- $r=10$ - range of a is $5 \le a \le 9$
- If $a=6$, $n=2$ - 133 numbers in range $\bar{6}\bar{6} \le X \le 66$
- 13 values for each digit - total of $13^2 = 169$ representations - 27% redundancy
- 1 has only one representation - 01 - $1\bar{9}$ is not valid
- 4 has two representations - 04 and $1\bar{6}$

Eliminating Carry Propagation Chains

- Calculating $(x_{n-1},...,x_0) \pm (y_{n-1},...,y_0)=(s_{n-1},...,s_0)$
- Breaking the carry chains - an algorithm in which sum digit s_i depends only on the four operand digits x_i, y_i, x_{i-1}, y_{i-1}
- Addition time independent of length of operands
- An algorithm that achieves this independence:
- **Step 1:** Compute interim sum u_i and carry digit c_i
- $u_i = x_i + y_i - r\,c_i$
  where

$$c_i = \begin{cases} 1 & \text{if } (x_i + y_i) \ge a \\ \bar{1} & \text{if } (x_i + y_i) \le \bar{a} \\ 0 & \text{if } |x_i + y_i| < a \end{cases}$$

- **Step 2**: Calculate the final sum $s_i = u_i + c_{i-1}$

Example - r=10, a=6

- $x_i=\bar{6},\dots,0,1,\dots,6$
- **Step 1** - $u_i = (x_i+y_i) - 10\,c_i$
- **Example** - 3645+1456

$$c_i = \begin{cases} 1 & \text{if } (x_i+y_i) \geq 6 \\ \bar{1} & \text{if } (x_i+y_i) \leq \bar{6} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{r} 3645 \\ +\,1456 \\ \hline 5101 \end{array}$$

$$\begin{array}{rccccl} & 3 & 6 & 4 & 5 & x \\ + & 1 & 4 & 5 & 6 & y \\ \hline 0 & 1 & 1 & 1 & & c \\ & 4 & 0 & \bar{1} & 1 & u \\ \hline & 5 & 1 & 0 & 1 & s \end{array}$$

- In conventional decimal number system - carry propagates from least to most significant digit
- Here - no carry propagation chain
- Carry bits shifted to left to simplify execution of second step

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Converting Representations

- This addition algorithm can be used for converting a decimal number to SD form by considering each digit as the sum x_i+y_i above
- **Example** - converting decimal 6849 to SD

$$\begin{array}{lccccc} x_i+y_i & & 6 & 8 & 4 & 9 \\ \hline c & 1 & 1 & 0 & 1 & \\ u & & \bar{4} & \bar{2} & 4 & \bar{1} \\ \hline s & 1 & \bar{3} & \bar{2} & 5 & \bar{1} \end{array}$$

- Converting SD to decimal - subtracting digits with negative weight from digits with positive weight
- **Example** - converting $1\bar{3}\bar{2}5\bar{1}$ to decimal

$$\begin{array}{r} 10050 \\ -03201 \\ \hline 6849 \end{array}$$

Proof of the Two-Step Algorithm

- To guarantee no new carry - $|s_i|$ £ a
- Since $|c_{i-1}|$£1, $|u_i|$ must be £ a-1 for all x_i,y_i
- Example - largest x_i+y_i is 2a Þ c_i=1, u_i=2a-r since a £ r-1, u_i=2a-r £ a-1
- Example - smallest x_i+y_i for which c_i=1 is a Þ u_i=a-r<0 or $|u_i|$=r-a; to get $|u_i|$ £ a-1, 2a ³ r+1
- Selected digit set must satisfy $\left\lceil \frac{r+1}{2} \right\rceil \leq a \leq r-1$
- Exercise - show that for all values of x_i+y_i, $|u_i|$ £ a-1 if $a \geq \left\lceil \frac{r+1}{2} \right\rceil$
- Example - for SD decimal numbers, a ³ 6 guarantees no new carries in previous algorithm

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Addition of Binary SD Numbers

- Only one possible digit set - $\{1,0,\bar{1}\}$ - a=1
- Interim sum and carry in addition algorithm -

$$u_i = (x_i + y_i) - 2c_i \qquad c_i = \begin{cases} 1 & \text{if } (x_i+y_i) \geq 1 \\ \bar{1} & \text{if } (x_i+y_i) \leq \bar{1} \\ 0 & \text{if } (x_i+y_i) = 0. \end{cases}$$

- Summary of rules -

x_iy_i	00	01	$0\bar{1}$	11	$\bar{1}\bar{1}$	$1\bar{1}$
c_i	0	1	$\bar{1}$	1	$\bar{1}$	0
u_i	0	$\bar{1}$	1	0	0	0

- Addition is commutative - $10,\bar{1}0,\bar{1}1$ not included
- In the binary case - $a \geq \left\lceil \frac{r+1}{2} \right\rceil = 2$ cannot be satisfied
- No guarantee that a new carry will not be generated in the second step of the algorithm

Addition - Carry Generation

- ◆ **If operands do not have $\bar{1}$ - new carries not generated**
- ◆ **Example -**
  - * In conventional representation - a carry propagates from least to most significant position
  - * Here - no carry propagation chain exists

$$\begin{array}{cccccccl}
 & & 1 & 1 & \cdots & 1 & 1 & \\
+ & & 0 & 0 & \cdots & 0 & 1 & \\
\hline
 & 1 & 1 & 1 & \cdots & 1 & & c_i \\
 & & \bar{1} & \bar{1} & \cdots & \bar{1} & 0 & u_i \\
\hline
 & 1 & 0 & 0 & \cdots & 0 & 0 & s_i
\end{array}$$

- ◆ **If operands have $\bar{1}$ - new carries may be generated**
- ◆ **Example -**
  - * If $x_{i-1}y_{i-1} = 01$ - $c_{i-1} = 1$ and if $x_iy_i = 0\bar{1}$ - $u_i = 1$ $s_i = u_i+c_{i-1} = 1+1$ - a new carry is generated
  - * Stars indicate positions where new carries are generated and must be allowed to propagate

$$\begin{array}{ccccccccl}
 & & 0 & \bar{1} & 1 & \bar{1} & 1 & 1 & \\
+ & & 1 & 0 & 0 & \bar{1} & 0 & 1 & \\
\hline
 & 1 & \bar{1} & 1 & \bar{1} & 1 & 1 & & c_i \\
 & & \bar{1} & 1 & \bar{1} & 0 & \bar{1} & 0 & u_i \\
\hline
 & & * & * & * & 1 & 0 & 0 & s_i
\end{array}$$

Addition - Avoiding Carry Generation

- ◆ $c_{i-1}=u_i=1$ when $x_iy_i = 0\bar{1}$ and $x_{i-1}y_{i-1} = 11$ or 01
- ◆ Selecting $c_i=0$ - $u_i = \bar{1}$
- ◆ However, for $x_{i-1}y_{i-1}=\bar{1}\bar{1}$ Þ $c_{i-1}=\bar{1}$ and we must still select $c_i=\bar{1}$, $u_i=1$
- ◆ Similarly, when $x_iy_i=01$ and $x_{i-1}y_{i-1}= \bar{1}\bar{1}$ or $0\bar{1}$, instead of $u_i=\bar{1}$ Þ select $c_i=0$ & $u_i=1$
- ◆ u_i and c_i can be determined by examining the two bits to the right $x_{i-1}y_{i-1}$
- ◆ u_i and c_i can still be calculated in parallel for all bit positions

x_iy_i	00	01	$0\bar{1}$	11	$\bar{1}\bar{1}$	$1\bar{1}$
c_i	0	1	$\bar{1}$	1	$\bar{1}$	0
u_i	0	$\bar{1}$	1	0	0	0

Addition of Binary SD Numbers - Modified Rules

$x_i y_i$	00	01	01	$0\bar{1}$	$0\bar{1}$	11	$\bar{1}\bar{1}$
$x_{i-1}\,y_{i-1}$	–	neither is $\bar{1}$	at least one is $\bar{1}$	neither is $\bar{1}$	at least one is $\bar{1}$	–	–
c_i	0	1	0	0	$\bar{1}$	1	$\bar{1}$
u_i	0	$\bar{1}$	1	$\bar{1}$	1	0	0

- **Example** -

$$\begin{array}{cccccccc}
 & 1 & \bar{1} & 0 & \bar{1} & 1 & & 7 \\
 & 0 & \bar{1} & 1 & 0 & 0 & & -4 \\
\hline
0 & \bar{1} & 0 & 0 & 1 & & & c \\
 & 1 & 0 & 1 & \bar{1} & \bar{1} & & u \\
\hline
 & 0 & 0 & 1 & 0 & \bar{1} & & 3
\end{array}$$

- Direct summation of the two operands results in $001\bar{1}1$, equivalent to 00101, all representing 3_{10}

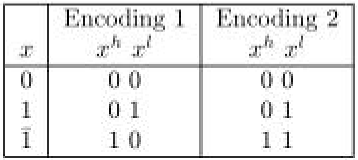
Minimal Representations of Binary SD Numbers

- Representation with a minimal number of nonzero digits
- Important for fast multiplication and division algorithms
- Nonzero digits - add/subtract operations, zero digits - shift-only operations
- **Example** - Representations of $X=7$:
- $100\bar{1}$ is the minimal representation
- The **canonical recoding algorithm** generates minimal SD representations of given binary numbers

	8	4	2	1
	0	1	1	1
	1	$\bar{1}$	1	1
	1	0	$\bar{1}$	1
	1	0	0	$\bar{1}$
1	$\bar{1}$	$\bar{1}$	1	1
		⋮		

Encoding of SD Binary Numbers

- 4x3x2=24 ways to encode 3 values of a binary signed bit x using 2 bits, x^h and x^l (high and low)
- Only nine are distinct encodings under permutation and logical negation - two have been used in practice
- Encoding #2 - a two's complement representation of the signed digit x
- Encoding #1 is sometimes preferable
  - Satisfies $x = x^l - x^h$ - 11 has a valid value of 0
  - Simplifies conversion from SD to two's complement - subtracting $x^h_{n-1}, x^h_{n-2}, \ldots, x^h_0$ from $x^l_{n-1}, x^l_{n-2}, \ldots, x^l_0$ using two's complement arithmetic
  - This requires a complete binary adder
  - A simpler conversion algorithm exists



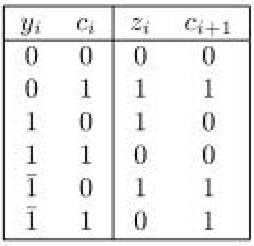
x	Encoding 1 x^h x^l	Encoding 2 x^h x^l
0	0 0	0 0
1	0 1	0 1
$\bar{1}$	1 0	1 1

Conversion Algorithm - Simpler Circuit

- Binary signed digits examined one at a time, right to left
- All occurrences of $\bar{1}$ are removed and the negative sign is forwarded to the most significant bit, the only bit with a negative weight in the two's complement representation
- The rightmost $\bar{1}$ is replaced by 1 and the negative sign is forwarded to the left, replacing 0's by 1's until a 1 is reached, which ``consumes" the negative sign and is replaced by 0
- If a 1 is not reached - the 0 in the most significant position is turned into a 1, becoming the negative sign bit of the two's complement representation
- If a second $\bar{1}$ is encountered before a 1 is, it is replaced by a 0 and the forwarding of the negative sign continues
- The negative sign is forwarded with the aid of a "borrow" bit which equals 1 as long as a $\bar{1}$ is being forwarded, and equals 0 otherwise

Conversion Algorithm Rules

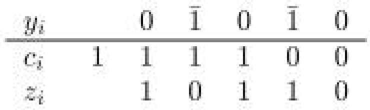
- y_i - i-th digit of the SD number
- z_i - i-th bit of the two's complement representation
- c_i - previous borrow
- c_{i+1} - next borrow
- For the least significant digit we assume $c_0=0$
- $y_i - c_i = z_i - 2c_{i+1}$



y_i	c_i	z_i	c_{i+1}
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	0
$\bar{1}$	0	1	1
$\bar{1}$	1	0	1

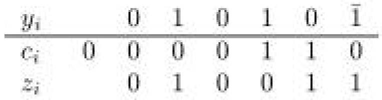
Conversion Algorithm - Cont.

- **Example** - -10_{10} - converting SD to two's complement



y_i		0	$\bar{1}$	0	$\bar{1}$	0
c_i	1	1	1	1	0	0
z_i		1	0	1	1	0

- Range of representable numbers in SD method - almost double that of the two's complement method
- n-digit SD number must be converted to an $(n+1)$-bit two's complement representation



y_i		0	1	0	1	0	$\bar{1}$
c_i	0	0	0	0	1	1	0
z_i		0	1	0	0	1	1

- Without the extra bit position - the number 19 would be converted to -13